Technical note accompanying

'Optimal Product Variety and Economic Growth:
The Trade-off between Internal and External Economies of Scale'

by Henri L.F. de Groot and R. Nahuis

This note discusses a generalization of the model presented in the paper 'Optimal Product Variety and Economic Growth: The Trade-off between Internal and External Economies of Scale' and briefly points at the implications of this generalization for the stability properties of the model. The model is similar to the model in the paper, except for the modelling of the engine of growth. More specifically, we generalize the engine of growth (equation 7) to

$$\dot{h}_i = \xi h_i L_{ri}^{\gamma}, \qquad \text{(N.1)}$$

where L_{ri} represents research labour, ξ is the research productivity parameter and γ captures the degree of diminishing returns to R&D labour.

Solution of the model

The solution procedure is as before and will now briefly be discussed (analogous to Appendix A in the paper). Firms in the assembly sector perform a two-stage maximization procedure. In the first stage, they solve

$$\max_{L_C, X} \Pi_C = CP_C - wL_C - P_x X, \qquad \text{(N.2)}$$

where Π_C denotes profits and P_X is the price index for the composite good. Optimization yields

$$\frac{\partial \Pi_C}{\partial X} = 0 \iff \beta CP_C = XP_X, \text{ and} \qquad \text{(N.3)}$$

$$\frac{\partial \Pi_C}{\partial L_C} = 0 \iff (1 - \beta)CP_C = L_C w, \qquad \text{(N.4)}$$

which is the standard Cobb-Douglas result of constant income shares.

The objective of firms in the intermediate goods sector can be written as

$$\max \int_0^{\infty} \left[x_i p_{xi} - (L_{xi} + L_{ri} + L_f) w \right] e^{-rt} dt. \qquad \text{(N.5)}$$

r is the interest rate at which firms could invest their money in the financial market. The current-value Hamiltonian corresponding to this dynamic optimization problem reads as (where, in contrast to the analysis in the main text, we allow for diminishing returns to research labour in knowledge accumulation ($\gamma \leq 1$))

$$H = x_i p_{xi} - (L_{xi}+L_{ri}+L_f)w + p_{hi}\xi h_i L_{ri}^{\gamma}, \tag{N.6}$$

where p_{hi} is the shadow price corresponding to knowledge. The first-order conditions to the inter-temporal optimization problem are (assuming symmetry so that we can drop the brand index i)

$$\frac{\partial H}{\partial L_x} = hp_x\left(1-\frac{1}{\varepsilon}\right) - w = 0 \quad \Leftrightarrow \quad p_x = \frac{\varepsilon}{\varepsilon-1}\frac{w}{h}, \tag{N.7}$$

according to which firms engage in mark-up pricing,

$$\frac{\partial H}{\partial L_r} = -w + p_h\xi h\gamma L_{ri}^{\gamma-1} = 0 \quad \Leftrightarrow \quad w = p_h\xi h\gamma L_{ri}^{\gamma-1}, \tag{N.8}$$

showing that firms allocate R&D labour to this sector as long as the marginal benefits of doing so ($p_h\xi h\gamma L_{ri}^{\gamma-1}$) exceed the marginal costs (w), and

$$-L_x p_x\frac{\varepsilon-1}{\varepsilon} - p_h\xi L_r^{\gamma} = \dot{p}_h - rp_h \quad \Leftrightarrow \quad \frac{\dot{p}_h}{p_h} + L_x\frac{p_x}{p_h}\frac{\varepsilon-1}{\varepsilon} + \xi L_r^{\gamma} = r, \tag{N.9}$$

which is the no-arbitrage condition. Using the expressions for the price of intermediates (equation (N.7)) and the allocation rule of research labour (equation (N.8)), we can derive

$$\frac{\dot{p}_x}{p_x} = \frac{\dot{w}}{w} - \frac{\dot{h}}{h} = \frac{\dot{w}}{w} - g = \frac{\dot{p}_h}{p_h} + (\gamma-1)\frac{\dot{L}_r}{L_r}. \tag{N.10}$$

Note that the wage rate is the numéraire (w=1). In the steady state, the allocation of labour is constant, and the number of firms is fixed. This gives rise to

$$g_C = \beta g_x = \beta g \quad \text{and} \quad \frac{\dot{P}_C}{P_C} = \beta\frac{\dot{p}_x}{p_x} = -\beta g. \tag{N.11}$$

So we can write the Ramsey rule as

$$r = \beta(\rho-1)g + \theta. \tag{N.12}$$

This equation, that corresponds to equation (10) in the main text of the paper, is drawn in figure A.1 and A.2 as the WW-curve. Using equations (N.7), (N.8), and (N.10), the no-arbitrage condition (N.9) yields

$$r = \xi L_x\gamma L_r^{\gamma-1}. \tag{N.13}$$

Imposing free entry and exit in the intermediate goods sector results in zero excess profits (equation (9) in the paper). Substituting the price for intermediates (equation (N.7)) and equation (6) in the paper into the zero-profit condition (equation (9) in the paper), it boils down to

$$\frac{L_x + L_r + L_f}{L_x} = \frac{\varepsilon}{\varepsilon - 1} \quad \Leftrightarrow \quad L_x = (\varepsilon - 1)(L_r + L_f). \tag{N.14}$$

The firm size in relation to the size of the production department is thus equal to the mark-up. The rate of return on investment can now be written as

$$r = (\varepsilon - 1)\left[\left(\frac{g}{\xi}\right)^{\frac{1}{\gamma}} + L_f\right]\xi\gamma\left(\frac{g}{\xi}\right)^{\frac{\gamma-1}{\gamma}}. \tag{N.15}$$

For γ=1, equation (10) in the main text of the paper results. This expression is drawn as the PP-curve in figure A.1 and A.2. Confronting the realized (see equation (N.15)) and required rate of return (given by equation (N.12)), yields the equilibrium growth and interest rate. In figure A.1, the planned and required rate of return have been depicted for the case in which γ=1. Stability of the interior solution with a positive growth rate requires that the realized rate of return intersects the required rate of return from above (see the discussion on stability in the paper and Evans et al., 1998). The intuition for this is that the cost of hiring more research labour (the required rate of return in terms of growth, equation (N.12)) should rise more rapidly then the benefits of hiring them (the realized rate of return, equation (N.15)). This holds if $\beta(\rho\text{-}1)>\varepsilon\text{-}1>\theta/\xi L_f$. The solution corresponds to equation (11) in the main text. If this restriction is not satisfied, corner solutions result, as we will show by discussing two alternative positions of the WW-curve in figure A.1. The special case of logarithmic utility (ρ=1; Case II in figure A.1) which violates the first part of the inequality would, for example, result in a corner solution with only one firm. Violation of the second part of the inequality - for example due to a very high subjective discount rate (Case I in figure A.1) - would result in zero growth. For a more detailed analysis of corner solutions in this type of model, we refer to Smulders (1994). Figure A.2 depicts a more general case of logarithmic utility and γ<1. In this case, a stable equilibrium exists at point E'. This result reveals that the strictness of the stability condition in case γ=1 can be circumvented by allowing for diminishing returns to research labour in knowledge accumulation.

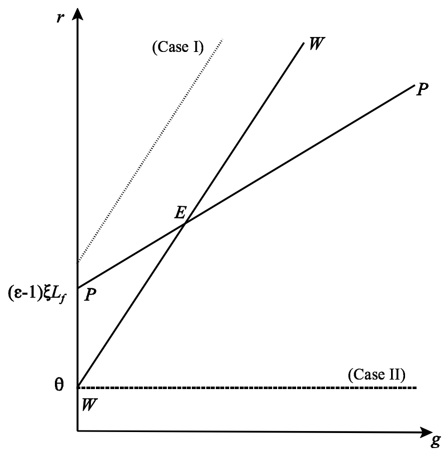

Figure A.1.
Equilibrium solution of the model (γ=1)

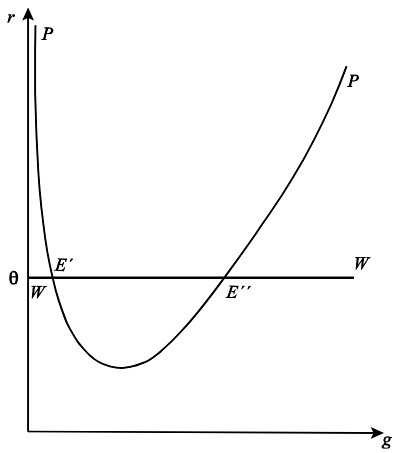

Figure A.2.
Equilibrium solution of the model (γ<1)

The paper and the remainder of this note restrict attention to the linear research technology and cases in which the imposed parameter restriction holds. The equilibrium number of high-tech firms and the allocation of labour can then be determined using labour market equilibrium (equation (8) in the paper), and

$$CP_C = \frac{Np_{xi}x_i}{\beta} = \frac{L_C w}{1-\beta} \quad \text{so} \quad \frac{NL_x \frac{\varepsilon}{\varepsilon-1}}{\beta} = \frac{L_C}{1-\beta}, \tag{N.16}$$

which shows how the assembly sector optimally chooses between labour and intermediates. This leaves us with

$$N = \frac{[\beta(\rho-1) - (\varepsilon-1)]\xi\beta L}{\varepsilon[\beta(\rho-1)\xi L_f - \theta]} \quad \text{and} \quad L_C = (1-\beta)L. \tag{N.17}$$

This equation corresponds to equation (12) in the main text of the paper.

The first-best social optimum

We now turn to the solution of the first-best social optimum, allowing for diminishing returns to research labour. The current-value Hamiltonian corresponding to the optimal-control problem given in section 4.1 of the paper is[1]

$$H = \frac{1}{1-\rho}\left[\left(L - N(L_x + L_r + L_f)\right)^{1-\beta}\left(N^\sigma h L_x\right)^\beta\right]^{1-\rho} + p_h \xi h L_r^\gamma. \tag{N.18}$$

The first-order conditions corresponding to this problem are

$$\frac{\partial H}{\partial L_x} = 0 \quad \Leftrightarrow \quad (1-\beta)NL_x = \beta L_C, \tag{N.19}$$

$$\frac{\partial H}{\partial L_r} = 0 \quad \Leftrightarrow \quad p_h = (1-\beta)\frac{C^{1-\rho}N}{L_C \xi h \gamma L_r^{\gamma-1}}, \tag{N.20}$$

$$\frac{\partial H}{\partial N} = 0 \quad \Leftrightarrow \quad L_C = \frac{(1-\beta)L}{\beta\sigma + 1 - \beta} = L_C^{FB}, \tag{N.21}$$

$$\dot{p}_h + \frac{\partial H}{\partial h} = \theta p_h \quad \Leftrightarrow \quad \dot{p}_h + \frac{C^{1-\rho}\beta}{h} + p_h \xi L_r^\gamma = \theta p_h. \tag{N.22}$$

From equation (N.21), it is evident that employment in the assembly sector is constant. Using equation (N.19), it then follows that NL_x is constant. Substitution of the expressions for NL_x and L_C in the labour market constraint (equation (8) in the paper) gives an expression for L_r and consequently for the optimal growth rate as given in the main text. We have restricted attention in the main text to the case in which

[1] Use that under symmetry $X = N^{\sigma-1}Nx = N^\sigma hL_x$.

γ=1.

Next, we turn to the determination of the optimal number of firms in the intermediate goods sector in the first-best social optimum. Differentiation of equation (N.20) with respect to time yields

$$\frac{\dot{p}_h}{p_h} = (1-\rho)\frac{\dot{C}}{C} + \frac{\dot{N}}{N} - \frac{\dot{L}_C}{L_C} - \frac{\dot{h}}{h} - (\gamma - 1)\frac{\dot{L}_r}{L_r}. \tag{N.23}$$

We know by combining the equations (N.20) and (N.22) that

$$\frac{\dot{p}_h}{p_h} = \theta - \xi L_r - \frac{\beta\xi L_C \gamma L_r^{\gamma-1}}{N(1-\beta)}. \tag{N.24}$$

By differentiation of equation (3) in the paper with respect to time, we get (using symmetry among the intermediate goods producers)

$$\frac{\dot{C}}{C} = \beta\frac{\dot{X}}{X} + (1-\beta)\frac{\dot{L}_C}{L_C} = \beta\left[(\sigma-1)\frac{\dot{N}}{N} + \frac{\dot{N}}{N} + \frac{\dot{h}}{h} + \frac{\dot{L}_x}{L_x}\right] + (1-\beta)\frac{\dot{L}_C}{L_C}. \tag{N.25}$$

We now use the fact that in the steady state the allocation of labour is constant and that labour-market equilibrium holds to arrive at

$$\frac{\xi\gamma(L_r+L_f)}{(\sigma-1)L_r^{1-\gamma}} = \theta + \beta(\rho-1)\xi L_r^{\gamma}. \tag{N.26}$$

This equation reflects that in the optimal allocation, the costs of investing in R&D (the right-hand side of equation (N.26)) should equal the benefits (the left-hand side of equation (N.26)). The left-hand side and right-hand side of this expression can be depicted in figures analogous to figures A.1 and A.2. In the special case in which γ=1, we need to impose $\beta(\rho-1)>1/(\sigma-1)>\theta/\xi L_f$ in order to avoid corner solutions. Note the similarity between this condition and the condition imposed in the market equilibrium. As before, allowing for diminishing returns to scale in R&D labour (i.e. the Inada conditions are fulfilled) loosens this restriction.

References

EVANS G.W., HONKAPOHJA S., ROMER P., 1998, Growth Cycles, *American Economic Review*, 88, 495-515.

GROOT H.L.F. DE, NAHUIS R., 2000, Optimal Product Variety and Economic Growth: The Trade-off between Internal and External Economies of Scale, paper under submission.

SMULDERS S., 1994, Growth, Market Structure and the Environment, Essays on the Theory of Endogenous Economic Growth, Ph.D. thesis, Tilburg University, Tilburg.